Int. J. Solids Structures Vol. 19, No. 8, pp. 677–692, 1983
Printed in Great Britain.

0020–7683/83 $3.00 + .00

VARIATIONAL APPROACH BY MEANS OF ADJOINT SYSTEMS TO STRUCTURAL OPTIMIZATION AND SENSITIVITY ANALYSIS—I

VARIATION OF MATERIAL PARAMETERS WITHIN FIXED DOMAIN

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(*Received 9 December 1981; in revised form 6 December 1982*)

Abstract—For a linear elastic structure, the first variation of an arbitrary stress, strain and displacement functionals corresponding to variation of material parameters within specified domain is derived by using the solution for primary and adjoint systems. This variation is of fundamental importance in sensitivity analysis, optimal design and identification problems. Simple examples of optimal stiffness design and identification of stiffness parameters in beams are presented.

1. INTRODUCTION

In many problems of structural mechanics there is a need to assess the effect of variation of one or several design functions, such as cross-sectional dimension, shape, support condition, on the stress or displacement states of a structure. To avoid numerous solutions of boundary-value problems for modified structural parameters, a variational procedure can be applied and the first or second variation of the respective functional can be obtained in the vicinity of a given state.

The present paper is concerned with such class of problems for which these variations can be explicitly expressed in terms of variations of design functions. The variations of arbitrary stress, strain or displacement functionals will be derived and expressed in terms of stress and strain fields of primary and adjoint structures. Our analysis will be limited to linearly elastic structures and first the variation of stiffness or cross-sectional moduli within fixed domain will be analysed. In the second part of this work (Part II) the variation of external or internal boundaries will be considered and the associated variations of respective functionals will be derived. The application to optimal design or identification problems will be presented in the last section of the paper. However the significance of the obtained results is much broader since the present approach constitutes the foundation for sensitivity analysis, optimal identification of material or shape parameters and for assessment of structure degradation.

2. VARIATION OF STIFFNESS OR COMPLIANCE MODULI WITHIN FIXED DOMAIN

Our analysis will extend the previous treatment of the same problem for surface structures, see Mróz and Mironov[1], and is related to variational approach to sensitivity analysis of structures, presented by Haug and Rousselet[2]. Consider first the case of a fixed domain of a structure for which the stress and displacement conditions are specified on the portions S_T and S_u of its boundary. Assume that stress and strain states are interrelated by the linear Hookes law†

$$\boldsymbol{\sigma} = \mathbf{D}(\varphi_k)\cdot\boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} = \mathbf{E}(\varphi_k)\cdot\boldsymbol{\sigma} \tag{1}$$

where the stiffness and compliance matrices **D** and **E** depend on a set of control functions

†The dot between two tensors of different orders denotes the summation with respect to indices of the tensor of lower order. Thus $\mathbf{D}\cdot\boldsymbol{\epsilon} = D_{ijkl}\epsilon_{kl}$, $\boldsymbol{\sigma}\cdot\mathbf{n} = \sigma_{ij}n_j$ and for two vectors **a** and **b**, $\mathbf{a}\cdot\mathbf{b} = a_ib_i$ denotes their scalar product.

$\varphi_k = \varphi_k(x)$. In particular, the functions φ_k can be identified with elastic moduli of the material.

Consider a small variation of φ_k and $\boldsymbol{\epsilon}$ and the associated variation of the stress state

$$\delta\boldsymbol{\sigma} = \mathbf{D}\cdot\delta\boldsymbol{\epsilon} + \delta\mathbf{D}\cdot\boldsymbol{\epsilon} = \mathbf{D}\cdot\delta\boldsymbol{\epsilon} + \frac{\partial\mathbf{D}}{\partial\varphi_k}\delta\varphi_k\cdot\delta\boldsymbol{\epsilon} = \delta\boldsymbol{\sigma}' + \delta\boldsymbol{\sigma}'' \tag{2}$$

where $\delta\boldsymbol{\sigma}'$ and $\delta\boldsymbol{\sigma}''$ are the stress variation components due to strain and stiffness matrix variations. For a uniaxial case, these variations are presented in Fig. 1. Assuming that the stress variation field $\delta\boldsymbol{\sigma}$ satisfies the equilibrium equations and boundary conditions

$$\operatorname{div}\delta\boldsymbol{\sigma} = 0 \quad \text{within } V, \quad \delta\boldsymbol{\sigma}\cdot\mathbf{n} = 0 \text{ on } S_T, \tag{3}$$

for any kinematically admissible strain field $\boldsymbol{\epsilon}$, the virtual work equation takes the form

$$\int \delta\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\,\mathrm{d}V = \int \delta\mathbf{T}\cdot\mathbf{u}^0\,\mathrm{d}S_u, \tag{4}$$

where $\mathbf{u}^0$ denote the specified displacements on S_u. Similarly, for the stress and compliance matrix variation, we have

$$\delta\boldsymbol{\epsilon} = \mathbf{E}\cdot\delta\boldsymbol{\sigma} + \delta\mathbf{E}\cdot\boldsymbol{\sigma} = \mathbf{E}\cdot\delta\boldsymbol{\sigma} + \frac{\partial\mathbf{E}}{\partial\varphi_k}\delta\varphi_k\cdot\boldsymbol{\sigma} = \delta\boldsymbol{\epsilon}' + \delta\boldsymbol{\epsilon}'' \tag{5}$$

where $\delta\boldsymbol{\epsilon}'$ corresponds to stress variation and $\delta\boldsymbol{\epsilon}''$ corresponds to compliance matrix variation, Fig. 1. Analogously to (4), for any statically admissible stress field, there is

$$\int \boldsymbol{\sigma}\cdot\delta\boldsymbol{\epsilon}\,\mathrm{d}V = \int \mathbf{T}^0\cdot\delta\mathbf{u}\,\mathrm{d}S_T + \int \mathbf{f}\cdot\delta\mathbf{u}\,\mathrm{d}V \tag{6}$$

where $\mathbf{T}^0$ denotes the specified traction vector on S_T and $\mathbf{f}$ denotes the body force vector.

2.1 *First variation of an arbitrary functional*

Consider the following functional

$$G_1 = \int \psi(\boldsymbol{\sigma}, \varphi_k)\,\mathrm{d}V + \int h(\mathbf{u}, \varphi_k)\,\mathrm{d}V + \int f(\mathbf{T})\,\mathrm{d}S_u + \int g(\mathbf{u})\,\mathrm{d}S_T \tag{7}$$

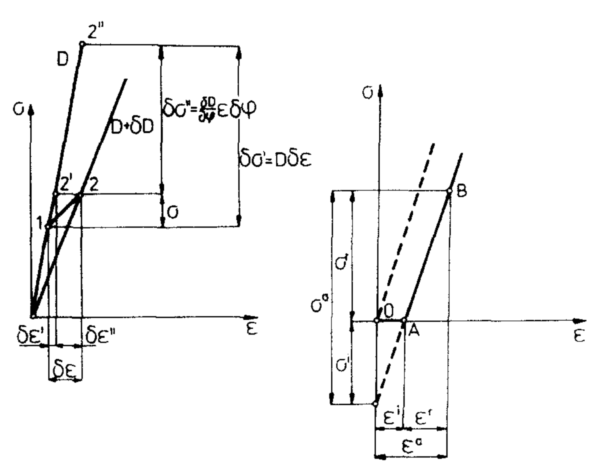

Fig. 1. (a) Variation of stress and strain due to stiffness modulus variation of the primary structure, (b) Stress and strain in the adjoint structure.

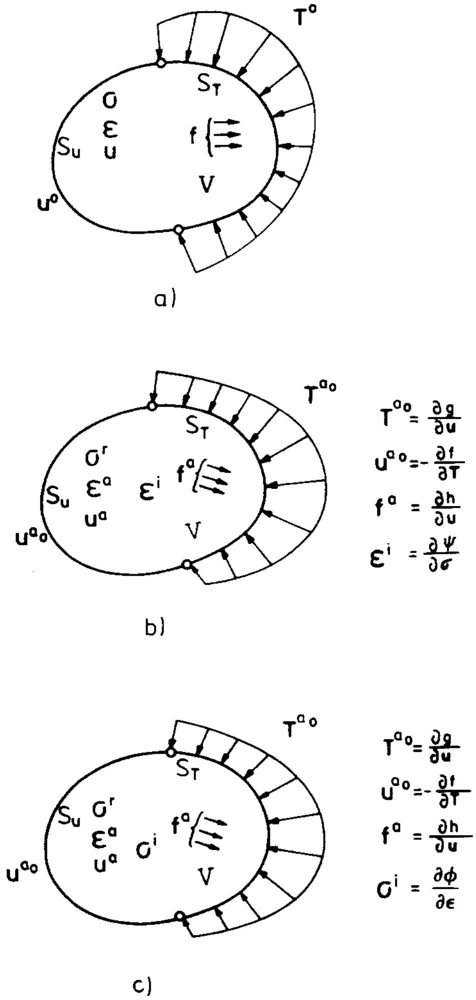

Fig. 2. (a) Primary structure subjected to variation of material parameters; Adjoint structures for stress, (b) and strain (c) functionals.

depending on stress and displacement fields within V, surface tractions on S_u, displacements on S_T and on control functions φ_k. When ψ, h, f and g are continuous and differentiable functions of their arguments, the first variation of G_1 equals

$$\delta G_1 = \int \frac{\partial \psi}{\partial \boldsymbol{\sigma}} \cdot \delta \boldsymbol{\sigma} \, \mathrm{d}V + \int \frac{\partial \psi}{\partial \varphi_k} \delta \varphi_k \, \mathrm{d}V + \int \frac{\partial h}{\partial \mathbf{u}} \cdot \delta \mathbf{u} \, \mathrm{d}V + \int \frac{\partial h}{\partial \varphi_k} \delta \varphi_k \, \mathrm{d}V + \int \frac{\partial f}{\partial \mathbf{T}} \cdot \delta \mathbf{T} \, \mathrm{d}S_u + \int \frac{\partial g}{\partial \mathbf{u}} \cdot \delta \mathbf{u} \, \mathrm{d}S_T. \tag{8}$$

To eliminate variations $\delta\boldsymbol{\sigma}$, $\delta\mathbf{T}$ and $\delta\mathbf{u}$ in (7), let us introduce the *adjoint structure* of the same form and material properties, but with the following boundary conditions

$$\mathbf{T}^{a_0} = \frac{\partial g}{\partial \mathbf{u}} \quad \text{on } S_T, \quad \mathbf{u}^{a_0} = -\frac{\partial f}{\partial \mathbf{T}} \text{on } S_u, \tag{9}$$

$$\mathbf{f}^a = \frac{\partial h}{\partial \mathbf{u}} \quad \text{within } V \tag{10}$$

and with the imposed field of initial strains $\boldsymbol{\epsilon}^i$ within V defined as follows

$$\boldsymbol{\epsilon}^i = \frac{\partial \psi}{\partial \boldsymbol{\sigma}} \quad \text{within } V. \tag{11}$$

Since this field does not satisfy the compatibility conditions, it will induce the residual stress field. Denoting the stress within the adjoint structure by $\boldsymbol{\sigma}^r$, its total strain field $\boldsymbol{\epsilon}^a$ can be presented as a sum, see Fig. 1(b),

$$\boldsymbol{\epsilon}^a = \boldsymbol{\epsilon}^i + \boldsymbol{\epsilon}^r \tag{12}$$

and is compatible with the displacement field $\mathbf{u}^a$ of this structure. The stress field $\boldsymbol{\sigma}^r$ is related to $\boldsymbol{\epsilon}^r$ by the Hooke's law

$$\boldsymbol{\sigma}^r = \mathbf{D}\cdot\boldsymbol{\epsilon}^r \tag{13}$$

and satisfies both the equilibrium and boundary conditions

$$\operatorname{div}\boldsymbol{\sigma}^r + \mathbf{f}^a = 0 \quad \text{within } V, \quad \boldsymbol{\sigma}^r\cdot\mathbf{n} = \mathbf{T}^{a_0} \text{ on } S_T. \tag{14}$$

in view of (2), (4), (11) and (12), the first term of (8) can be retransformed as follows

$$\int \frac{\partial\psi}{\partial\boldsymbol{\sigma}}\cdot\delta\boldsymbol{\sigma}\,\mathrm{d}V = \int \boldsymbol{\epsilon}^i\cdot\delta\boldsymbol{\sigma}\,\mathrm{d}V = \int (\boldsymbol{\epsilon}^a - \boldsymbol{\epsilon}^r)\cdot\delta\boldsymbol{\sigma}\,\mathrm{d}V = \int \mathbf{u}^{a_0}\cdot\delta\mathbf{T}\,\mathrm{d}S_u$$

$$-\int \boldsymbol{\epsilon}^r\cdot\delta\boldsymbol{\sigma}\,\mathrm{d}V = \int \mathbf{u}^{a_0}\cdot\delta\mathbf{T}\,\mathrm{d}S_u - \int \boldsymbol{\epsilon}^r\cdot\mathbf{D}\cdot\delta\boldsymbol{\epsilon}\,\mathrm{d}V - \int \boldsymbol{\epsilon}^r\cdot\frac{\partial\mathbf{D}}{\partial\varphi_k}\cdot\boldsymbol{\epsilon}\delta\varphi_k\,\mathrm{d}V \tag{15}$$

and since the reciprocity theorem is valid, we have

$$\int \boldsymbol{\epsilon}^r\cdot\mathbf{D}\cdot\delta\boldsymbol{\epsilon}\,\mathrm{d}V = \int \mathbf{D}\cdot\boldsymbol{\epsilon}^r\cdot\delta\boldsymbol{\epsilon}\,\mathrm{d}V = \int \boldsymbol{\sigma}^r\cdot\delta\boldsymbol{\epsilon}\,\mathrm{d}V = \int \mathbf{T}^{a_0}\cdot\delta\mathbf{u}\,\mathrm{d}S_T + \int \mathbf{f}^a\cdot\delta\mathbf{u}\,\mathrm{d}V \tag{16}$$

and the first variation of G_1, that is eqn (8) is expressed as follows

$$\delta G_1 = \int \left(\frac{\partial\psi}{\partial\varphi_k} + \frac{\partial h}{\partial\varphi_k} - \boldsymbol{\epsilon}^r\cdot\frac{\partial\mathbf{D}}{\partial\varphi_k}\cdot\boldsymbol{\epsilon}\right)\delta\varphi_k\,\mathrm{d}V. \tag{17}$$

It is seen that δG_1 is expressed in terms of strain fields $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}^r$ of the primary and adjoint structures and is explicitly related to the variations of control functions.

In particular, when $f(\mathbf{T}) = 0$ on S_u, $h(\mathbf{u}) = 0$ within V and $g(\mathbf{u}) = 0$ on S_T, the functional (7) takes the form

$$G_1 = \int \psi(\boldsymbol{\sigma}, \varphi_k)\,\mathrm{d}V \tag{18}$$

and the boundary conditions for the adjoint structure are

$$\mathbf{T}^{a_0} = 0 \text{ on } S_T, \quad \mathbf{u}^{a_0} = 0 \text{ on } S_u, \tag{19}$$

whereas the initial strain field $\boldsymbol{\epsilon}^i$ remains the same. In this case $\boldsymbol{\sigma}^r$ and $\boldsymbol{\epsilon}^r$ are the residual stress and strain fields within the adjoint structure. On the other hand, when (7) becomes

$$G_1 = \int f(\mathbf{T})\,\mathrm{d}S_u + \int g(\mathbf{u})\,\mathrm{d}S_T, \tag{20}$$

its variation is expressed as follows

$$\delta G_1 = -\int \boldsymbol{\epsilon}^r\cdot\frac{\partial\mathbf{D}}{\partial\varphi_k}\,\delta\varphi_k\cdot\boldsymbol{\epsilon}\,\mathrm{d}V \tag{21}$$

and the adjoint structure satisfies the boundary conditions (9) with vanishing initial strains and body forces within V.

An alternative formulation can be presented when the objective functional depends on the strain field $\boldsymbol{\epsilon}$, displacement field $\mathbf{u}$ within V and on S_T, traction field $\mathbf{T}$ on S_u as well as on control functions φ_k, that is

$$G_2 = \int \phi(\boldsymbol{\epsilon}, \varphi_k)\,\mathrm{d}V + \int h(\mathbf{u}, \varphi_k)\,\mathrm{d}V + \int g(\mathbf{u})\,\mathrm{d}S_T + \int f(\mathbf{T})\,\mathrm{d}S_u \tag{22}$$

where ϕ, f, g and h are continuous and differentiable functions of their arguments. The first variation of (22) equals

$$\delta G_2 = \int \frac{\partial \phi}{\partial \boldsymbol{\epsilon}}\cdot\delta\boldsymbol{\epsilon}\,\mathrm{d}V + \int \frac{\partial \phi}{\partial \varphi_k}\delta\varphi_k\,\mathrm{d}V + \int \frac{\partial h}{\partial \mathbf{u}}\cdot\delta\mathbf{u}\,\mathrm{d}V + \int \frac{\partial h}{\partial \varphi_k}\delta\varphi_k\,\mathrm{d}V + \int \frac{\partial g}{\partial \mathbf{u}}\cdot\delta\mathbf{u}\,\mathrm{d}S_T + \int \frac{\partial f}{\partial \mathbf{T}}\cdot\delta\mathbf{T}\,\mathrm{d}S_u. \tag{23}$$

Let us introduce now the adjoint structure satisfying the boundary conditions (9) and (10), and with the imposed initial stress field

$$\boldsymbol{\sigma}^i = \frac{\partial \phi}{\partial \boldsymbol{\epsilon}} \quad \text{within } V, \tag{24}$$

for which the corresponding initial strain field is $\boldsymbol{\epsilon}^i = \mathbf{E}\cdot\boldsymbol{\sigma}^i$, see Fig. 1(b). The stress field within the adjoint structure is $\boldsymbol{\sigma}^r$ with the associated strains $\boldsymbol{\epsilon}^r = \mathbf{E}\cdot\boldsymbol{\sigma}^r$ and the total strain and displacement fields are $\boldsymbol{\epsilon}^a$, $\mathbf{u}^a$. The 'total' stress $\boldsymbol{\sigma}^a$ of the adjoint structure equals

$$\boldsymbol{\sigma}^a = \mathbf{D}\cdot\boldsymbol{\epsilon}^a = \boldsymbol{\sigma}^i + \boldsymbol{\sigma}^r. \tag{25}$$

Obviously, the stress field $\boldsymbol{\sigma}^r$ satisfies the equilibrium and boundary conditions

$$\operatorname{div}\boldsymbol{\sigma}^r + \mathbf{f}^a = 0 \quad \text{within } V, \quad \boldsymbol{\sigma}^r\cdot\mathbf{n} = \mathbf{T}^{a_0} \text{ on } S_T, \tag{26}$$

and $\mathbf{u}^a = \mathbf{u}^{a_0}$ on S_u. In view of (24)–(26), the first term of (23) can be retransformed as follows

$$\int \frac{\partial \phi}{\partial \boldsymbol{\epsilon}}\cdot\delta\boldsymbol{\epsilon}\,\mathrm{d}V = \int \boldsymbol{\sigma}^i\cdot\delta\boldsymbol{\epsilon}\,\mathrm{d}V = \int (\boldsymbol{\sigma}^a - \boldsymbol{\sigma}^r)\cdot\delta\boldsymbol{\epsilon}\,\mathrm{d}V = \int \boldsymbol{\sigma}^a\cdot\delta\boldsymbol{\epsilon}\,\mathrm{d}V$$

$$-\int \mathbf{T}^{a_0}\cdot\delta\mathbf{u}\,\mathrm{d}S_T = \int \boldsymbol{\sigma}^a\cdot\mathbf{E}\cdot\delta\boldsymbol{\sigma}\,\mathrm{d}V + \boldsymbol{\sigma}^a\cdot\frac{\partial \mathbf{E}}{\partial \varphi_k}\delta\varphi_k\cdot\boldsymbol{\sigma}\,\mathrm{d}V - \int \mathbf{T}^{a_0}\cdot\delta\mathbf{u}\,\mathrm{d}S_T \tag{27}$$

and since

$$\int \boldsymbol{\sigma}^a\cdot\mathbf{E}\cdot\delta\boldsymbol{\sigma}\,\mathrm{d}V = \int \mathbf{E}\cdot\boldsymbol{\sigma}^a\cdot\delta\boldsymbol{\sigma}\,\mathrm{d}V = \int \boldsymbol{\epsilon}^a\cdot\delta\boldsymbol{\sigma}\,\mathrm{d}V = \int \mathbf{u}^{a_0}\cdot\delta\mathbf{T}\,\mathrm{d}S_u, \tag{28}$$

the variation δG_2 by virtue of (9), (10) and (27), (28) equals

$$\delta G_2 = \int \left(\frac{\partial \phi}{\partial \varphi_k} + \frac{\partial h}{\partial \varphi_k} + \boldsymbol{\sigma}^a\cdot\frac{\partial \mathbf{E}}{\partial \varphi_k}\cdot\boldsymbol{\sigma}\right)\delta\varphi_k\,\mathrm{d}V \tag{29}$$

where $\boldsymbol{\sigma}$ is the actual stress field of the primary structure and $\boldsymbol{\sigma}^a$ is the 'total' stress field within the adjoint structure associated by the Hooke's law $\boldsymbol{\sigma}^a = \mathbf{D}\cdot\boldsymbol{\epsilon}^a$ with the compatible strain field $\boldsymbol{\epsilon}^a$.

It is easy to show that the expressions (17) and (29) are equivalent provided

$$\psi(\boldsymbol{\sigma}, \varphi_k) = \phi(\boldsymbol{\epsilon}, \varphi_k). \tag{30}$$

In fact, we have

$$\frac{\partial \psi}{\partial \boldsymbol{\sigma}} = \boldsymbol{\epsilon}^i = \mathbf{E}\cdot\boldsymbol{\sigma}^i = \mathbf{E}\cdot\frac{\partial \phi}{\partial \boldsymbol{\epsilon}} \tag{31}$$

and

$$\frac{\partial \psi}{\partial \varphi_k}\delta\varphi_k = \left(\frac{\partial \phi}{\partial \varphi_k}\delta\varphi_k\right)_{\epsilon} + \frac{\partial \phi}{\partial \boldsymbol{\epsilon}}\cdot(\delta\boldsymbol{\epsilon})_{\sigma} \tag{32}$$

where $(\ )_{\sigma}$ and $(\ )_{\epsilon}$ denote values for constant $\boldsymbol{\sigma}$ or $\boldsymbol{\epsilon}$, respectively. Since $(\delta\boldsymbol{\epsilon})_{\sigma}$ is the value of $\delta\boldsymbol{\epsilon}$ for $\delta\boldsymbol{\sigma} = 0$, from (2) it follows that

$$(\delta\boldsymbol{\epsilon})_{\sigma} = -\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial \varphi_k}\delta\varphi_k\cdot\boldsymbol{\epsilon}. \tag{33}$$

Substituting (33) into (32) and using (24), (25), it is obtained

$$\frac{\partial \psi}{\partial \varphi_k} = \frac{\partial \phi}{\partial \varphi_k} - \boldsymbol{\sigma}^a\cdot\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial \varphi_k}\cdot\mathbf{E}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}^r\cdot\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial \varphi_k}\cdot\boldsymbol{\epsilon} \tag{34}$$

and since $\mathbf{D}\cdot\mathbf{E} = \mathbf{I}$, there is

$$\frac{\partial \mathbf{D}}{\partial \varphi_k}\cdot\mathbf{E} = -\mathbf{D}\cdot\frac{\partial \mathbf{E}}{\partial \varphi_k},\quad \frac{\partial \mathbf{D}}{\partial \varphi_k} = -\mathbf{D}\cdot\frac{\partial \mathbf{E}}{\partial \varphi_k}\cdot\mathbf{D}. \tag{35}$$

Substituting (35) into (34), we have

$$\frac{\partial \psi}{\partial \varphi_k} - \boldsymbol{\epsilon}^r\cdot\frac{\partial \mathbf{D}}{\partial \varphi_k}\cdot\boldsymbol{\epsilon} = \frac{\partial \phi}{\partial \varphi_k} + \boldsymbol{\sigma}^a\cdot\frac{\partial \mathbf{E}}{\partial \varphi_k}\cdot\boldsymbol{\sigma} \tag{36}$$

that is, equivalence of the expressions (17) and (29). It is seen that the first variations of functionals G_1 and G_2 can be expressed in terms of strain or stress fields of primary and adjoint structures.

Let us note that the derived expressions can be applied in the case of beam of surface structures, such as plates or shells for which the shape of the median surface is specified but the thickness is allowed to vary, see [1]. The control function can then be identified with thickness and the variation of any stress or displacement functional due to thickness variation is expressed by (17) or (29) with proper replacement of stress and strain by generalized stresses and strains.

2.2 *Second variation of an arbitrary functional*

The expressions (17) and (29) for first variations of G_1 and G_2 can now be used in deriving the second variations. Consider first the functional G_1. From (17), it follows that

$$\begin{aligned}\delta^2 G_1 = \delta(\delta G_1) = \int\Bigg[\bigg[&\frac{\partial^2 \psi}{\partial \varphi_k \partial \boldsymbol{\sigma}}\cdot\delta\boldsymbol{\sigma} + \frac{\partial^2 \psi}{\partial \varphi_k \partial \varphi_l}\delta\varphi_l + \frac{\partial^2 h}{\partial \varphi_k \partial \mathbf{u}}\cdot\delta\mathbf{u}\\ &+ \frac{\partial^2 h}{\partial \varphi_k \partial \varphi_l}\delta\varphi_l + \delta\boldsymbol{\sigma}^r\cdot\frac{\partial \mathbf{E}}{\partial \varphi_k}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}^r\cdot\frac{\partial \mathbf{E}}{\partial \varphi_k}\cdot\delta\boldsymbol{\sigma}\\ &+ \boldsymbol{\sigma}^r\cdot\frac{\partial^2 \mathbf{E}}{\partial \varphi_k \partial \varphi_l}\delta\varphi_l\cdot\boldsymbol{\sigma}\bigg]\delta\varphi_k\,\mathrm{d}V.\end{aligned} \tag{37}$$

Using the virtual work equations

$$\int \delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\epsilon}\,\mathrm{d}V = \int \delta\mathbf{f}^a \cdot \delta\mathbf{u}\,\mathrm{d}V + \int \delta\mathbf{T}^{a_0} \cdot \delta\mathbf{u}\,\mathrm{d}S_T \tag{38}$$

and

$$\int \delta\boldsymbol{\sigma} \cdot \delta\boldsymbol{\epsilon}^a\,\mathrm{d}V = \int \delta\mathbf{T} \cdot \delta\mathbf{u}^{a_0}\,\mathrm{d}S_u \tag{39}$$

where

$$\delta\mathbf{f}^a = \delta\left(\frac{\partial h}{\partial \mathbf{u}}\right) = \frac{\partial^2 h}{\partial \mathbf{u}\partial \mathbf{u}} \cdot \delta\mathbf{u}, + \frac{\partial^2 h}{\partial u \partial \varphi_k}\delta\varphi_k \cdot \delta\mathbf{T}^{a_0} = \delta\left(\frac{\partial g}{\partial u}\right) = \frac{\partial^2 g}{\partial u \partial u} \cdot \delta\mathbf{u},$$

$$\delta\mathbf{u}^{a_0} = \delta\left(-\frac{\partial f}{\partial \mathbf{T}}\right) = -\frac{\partial^2 f}{\partial \mathbf{T}\partial \mathbf{T}} \cdot \delta\mathbf{T}, \delta\boldsymbol{\epsilon}^i = \delta\left(\frac{\partial \psi}{\partial \boldsymbol{\sigma}}\right) = \frac{\partial^2 \psi}{\partial \boldsymbol{\sigma}\partial \boldsymbol{\sigma}} \cdot \delta\boldsymbol{\sigma} + \frac{\partial^2 \psi}{\partial \boldsymbol{\sigma}\partial \varphi_k}\delta\varphi_k \tag{40}$$

$$\delta\boldsymbol{\epsilon}^a = \delta\boldsymbol{\epsilon}^r + \delta\boldsymbol{\epsilon}^i = \mathbf{E} \cdot \delta\boldsymbol{\sigma}^r + \frac{\partial \mathbf{E}}{\partial \varphi_k}\delta\varphi_k \cdot \boldsymbol{\sigma}^r + \delta\boldsymbol{\epsilon}^i,$$

the two terms occurring in (37) can be retransformed as follows

$$\int \frac{\partial^2 \psi}{\partial \boldsymbol{\sigma}\partial \varphi_k}\delta\varphi_k \cdot \delta\boldsymbol{\sigma}\,\mathrm{d}V = \int \left[-\mathbf{E} \cdot \delta\boldsymbol{\sigma} \cdot \delta\boldsymbol{\sigma}^r - \delta\boldsymbol{\sigma} \cdot \frac{\partial^2 \psi}{\partial \boldsymbol{\sigma}\partial \boldsymbol{\sigma}} \cdot \delta\boldsymbol{\sigma}\right.$$

and

$$\left. -\boldsymbol{\sigma}^r \cdot \frac{\partial \mathbf{E}}{\partial \varphi_k} \cdot \delta\boldsymbol{\sigma}\delta\varphi_k \right]\mathrm{d}V - \int \delta\mathbf{T} \cdot \frac{\partial^2 f}{\partial \mathbf{T}\partial \mathbf{T}} \cdot \delta\mathbf{T}\,\mathrm{d}S_u, \tag{41}$$

$$\int \boldsymbol{\sigma} \cdot \frac{\partial \mathbf{E}}{\partial \varphi_k} \cdot \partial\boldsymbol{\sigma}^r \delta\varphi_k\,\mathrm{d}V = \int \left[-\mathbf{E} \cdot \delta\boldsymbol{\sigma} \cdot \delta\boldsymbol{\sigma}^r + \frac{\partial^2 h}{\partial \mathbf{u}\partial \mathbf{u}} \cdot \delta\mathbf{u} \cdot \delta\mathbf{u} + \frac{\partial^2 h}{\partial \mathbf{u}\delta\varphi_k} \cdot \delta\mathbf{u}\delta\varphi_k \right]\mathrm{d}V$$
$$+ \int \frac{\partial^2 g}{\partial \mathbf{u}\partial \mathbf{u}} \cdot \delta\mathbf{u} \cdot \delta\mathbf{u}\,\mathrm{d}S_T. \tag{42}$$

The expression for second variation of G_1 can thus be presented in the form

$$\delta^2 G_1 = \int \left[-\frac{\partial^2 \psi}{\partial \boldsymbol{\sigma}\partial \boldsymbol{\sigma}} \cdot \delta\boldsymbol{\sigma} \cdot \delta\boldsymbol{\sigma} - 2\mathbf{E} \cdot \delta\boldsymbol{\sigma} \cdot \delta\boldsymbol{\sigma}^r + \frac{\partial^2 h}{\partial \mathbf{u}\partial \mathbf{u}} \cdot \delta\mathbf{u} \cdot \delta\mathbf{u}\right.$$

$$\left. + 2\frac{\partial^2 h}{\partial \mathbf{u}\partial \varphi_k} \cdot \delta\mathbf{u}\delta\varphi_k + \left(\frac{\partial^2 \psi}{\partial \varphi_k \partial \varphi_1} + \boldsymbol{\sigma}^r \cdot \frac{\partial^2 \mathbf{E}}{\partial \varphi_k \partial \varphi_1} \cdot \boldsymbol{\sigma}\right)\delta\varphi_k\delta\varphi_1 \right]\mathrm{d}V +$$

$$+ \int \frac{\partial^2 g}{\partial \mathbf{u}\partial \mathbf{u}} \cdot \delta\mathbf{u} \cdot \delta\mathbf{u}\,\mathrm{d}S_T - \int \frac{\partial^2 f}{\partial \mathbf{T}\partial \mathbf{T}} \cdot \delta\mathbf{T} \cdot \delta\mathbf{T}\,\mathrm{d}S_u. \tag{43}$$

Consider now the particular case when the functional G_1 coincides with the complementary energy, that is

$$G_1 = \Pi_\sigma = \int W(\boldsymbol{\sigma}, \varphi_k)\,\mathrm{d}V - \int \mathbf{T} \cdot \mathbf{u}^0\,\mathrm{d}S_u \tag{44}$$

where $W(\boldsymbol{\sigma})$ denotes the specific stress energy. Comparing (44) with the general expression (7), it is seen that

$$\psi = W(\boldsymbol{\sigma}, \varphi_k), \quad h = 0, \quad f(\mathbf{T}) = -\mathbf{T} \cdot \mathbf{u}^0, \quad g = 0 \tag{45}$$

and

$$\boldsymbol{\epsilon}^i = \frac{\partial W}{\partial \boldsymbol{\sigma}} = \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon}^r = 0, \quad \boldsymbol{\epsilon}^a = \boldsymbol{\epsilon}, \quad \mathbf{u}^{a_0} = \mathbf{u}^0$$

$$\mathbf{f}^a = 0, \quad \boldsymbol{\sigma}^r = 0, \quad \mathbf{T}^{a_0} = 0, \tag{46}$$

that is the adjoint structure, Fig. 3, is characterized by the same displacement and strain fields as the primary structure and vanishing stress field $\boldsymbol{\sigma}^r$.

In view of (17) and (46) the first variation of Π_σ now equals

$$\delta G_1 = \delta \Pi_\sigma = \int \frac{\partial W}{\partial \varphi_k} \delta\varphi_k \, dV \tag{47}$$

and the second variation is expressed as follows

$$\delta^2 G_1 = \delta^2 \Pi_\sigma = \int \left(-\frac{\partial^2 W}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\sigma}} \cdot \delta\boldsymbol{\sigma} \cdot \delta\boldsymbol{\sigma} + \frac{\partial^2 W}{\partial \varphi_k \partial \varphi_l} \delta\varphi_k \delta\varphi_l \right) dV \tag{48}$$

that is by the sum of two quadratic forms of $\delta\boldsymbol{\sigma}$ and $\delta\varphi_k$. The expression for the second

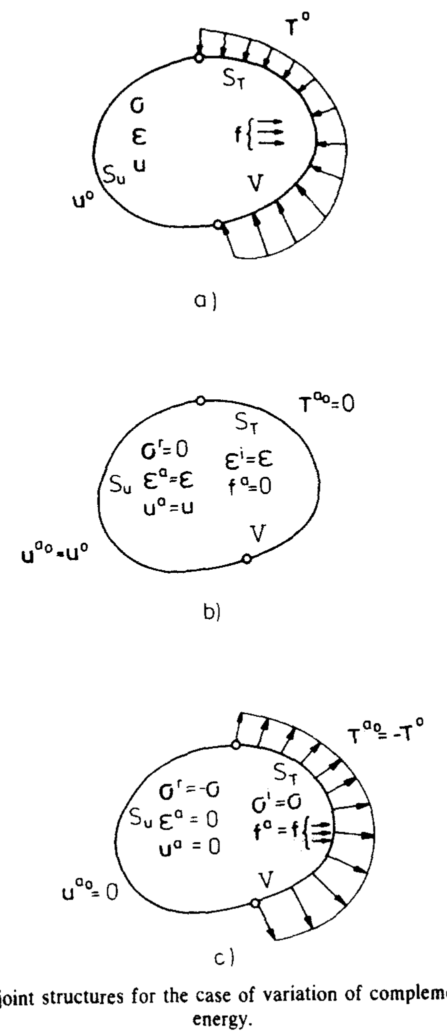

Fig. 3. Primary (a) and adjoint structures for the case of variation of complementary (b) and potential (c) energy.

variation of G_2 is expressed similarly to (43), namely

$$\delta^2 G_2 = \int \Big[-\frac{\partial^2 \phi}{\partial \boldsymbol{\epsilon} \partial \boldsymbol{\epsilon}} \cdot \delta\boldsymbol{\epsilon} \cdot \delta\boldsymbol{\epsilon} + 2\mathbf{D} \cdot \delta\boldsymbol{\epsilon} \cdot \delta\boldsymbol{\epsilon}^a - \frac{\partial^2 h}{\partial \mathbf{u} \partial \mathbf{u}} \cdot \delta\mathbf{u} \cdot \delta\mathbf{u} + \\ + \Big(\frac{\partial^2 \phi}{\partial \varphi_k \partial \varphi_1} - \boldsymbol{\epsilon}^a \cdot \frac{\partial^2 \mathbf{D}}{\partial \varphi_k \partial \varphi_1} \cdot \boldsymbol{\epsilon} \Big) \delta\varphi_k \delta\varphi_1 \Big] \,\mathrm{d}V + \\ - \int \frac{\partial^2 g}{\partial \mathbf{u} \partial \mathbf{u}} \cdot \delta\mathbf{u} \cdot \delta\mathbf{u} \,\mathrm{d}S_T + \int \frac{\partial^2 f}{\partial \mathbf{T} \partial \mathbf{T}} \cdot \delta\mathbf{T} \cdot \delta\mathbf{T} \,\mathrm{d}S_u, \tag{49}$$

where it was assumed that $h = h(\mathbf{u})$, that is independence of the function h on design variables φ_k. Consider now the particular case when G_2 coincides with the potential energy of the structure, thus

$$G_2 = \Pi_u = \int U(\boldsymbol{\epsilon}, \varphi_k) \,\mathrm{d}V - \int \mathbf{f} \cdot \mathbf{u} \,\mathrm{d}V - \int \mathbf{T}^0 \cdot \mathbf{u} \,\mathrm{d}S_T \tag{50}$$

where $U(\boldsymbol{\epsilon}, \varphi_k)$ denotes the specific strain energy. Since now, we have

$$\phi = U(\boldsymbol{\epsilon}, \varphi_k), \quad h = -\mathbf{f} \cdot \mathbf{u}, \quad g = -\mathbf{T}^0 \cdot \mathbf{u}, \quad f = 0 \tag{51}$$

and

$$\boldsymbol{\sigma}^i = \frac{\partial U}{\partial \boldsymbol{\epsilon}} = \boldsymbol{\sigma}, \quad \boldsymbol{\sigma}^r = -\boldsymbol{\sigma}, \quad \boldsymbol{\sigma}^a = 0, \quad \mathbf{u}^{a_0} = 0 \\ \mathbf{f}^a = \mathbf{f}, \quad \boldsymbol{\epsilon}^r = -\boldsymbol{\epsilon}, \quad \mathbf{u}^a = \boldsymbol{\epsilon}^a = 0, \quad \mathbf{T}^{a_0} = -\mathbf{T}^0, \tag{52}$$

in view of (29) and (49) the expression for the first and the second variations of Π_u are

$$\delta \Pi_u = \delta G_2 = \int \frac{\partial U}{\partial \varphi_k} \delta\varphi_k \,\mathrm{d}V, \tag{53}$$

and

$$\delta^2 \Pi_u = \delta^2 G_2 = \int \Big(-\frac{\partial^2 U}{\partial \boldsymbol{\epsilon} \partial \boldsymbol{\epsilon}} \cdot \delta\boldsymbol{\epsilon} \cdot \delta\boldsymbol{\epsilon} + \frac{\partial^2 U}{\partial \varphi_k \partial \varphi_1} \delta\varphi_k \delta\varphi_1 \Big) \,\mathrm{d}V \tag{54}$$

Let us note that now the displacement and strain fields within the adjoint structure vanish whereas the stress field is the same as in the primary structure. The expressions (53) and (54) were already derived in [5] in a different manner.

3. OPTIMALITY CONDITIONS FOR OPTIMAL DESIGN AND IDENTIFICATION PROBLEMS

The typical problem of optimal design is formulated as follows: minimize the cost of the structure with imposed behaviour constraints expressed in terms of stress, strain or displacement fields, thus

$$J = \int F(\varphi_k) \,\mathrm{d}V \to \min_\varphi \tag{55}$$

subject to the global stress constraint

$$H_1 = \int K(\boldsymbol{\sigma}, \varphi_k) \,\mathrm{d}V - H_0 \leq 0 \tag{56}$$

or displacement constraint

$$H_2 = \int L_1(\mathbf{u}, \varphi_k)\,\mathrm{d}V + \int L_2(\mathbf{u})\,\mathrm{d}S_T - L_0 \leq 0 \tag{57}$$

and other geometric shape constraints which will not be considered here. The functional G_1 now takes the form

$$G_1 = \int F(\varphi_k)\,\mathrm{d}V + \lambda\left[\int K(\boldsymbol{\sigma}, \varphi_k)\,\mathrm{d}V - H_0\right] \tag{58}$$

or

$$G_1' = \int F(\varphi_k)\,\mathrm{d}V + \lambda'\left[\int L_1(\mathbf{u}, \varphi_k)\,\mathrm{d}V + \int L_2(\mathbf{u})\,\mathrm{d}S_T - L_0\right] \tag{59}$$

where λ denotes the Lagrange multiplier and H_0, L_0 are specified parameters. In view of (17), the first variations of (58) and (59) now equal

$$\delta G_1 = \int\left[\frac{\partial F}{\partial \varphi_k} + \lambda\left(\frac{\partial K}{\partial \varphi_k} - \boldsymbol{\epsilon}^r \cdot \frac{\partial \mathbf{D}}{\partial \varphi_k} \cdot \boldsymbol{\epsilon}\right)\right]\delta\varphi_k\,\mathrm{d}V + \delta\lambda H_1 \tag{60}$$

and

$$\delta G_1' = \int\left[\frac{\partial F}{\partial \varphi_k} + \lambda'\left(\frac{\partial L_1}{\partial \varphi_k} - \boldsymbol{\epsilon}^r \cdot \frac{\partial \mathbf{D}}{\partial \varphi_k} \cdot \boldsymbol{\epsilon}\right)\right]\delta\varphi_k\,\mathrm{d}V + \delta\lambda' H_2. \tag{61}$$

The stationarity condition $\delta G_1 = 0$ provides the relations

$$\frac{\partial F}{\partial \varphi_k} = -\lambda\left[\frac{\partial K}{\partial \varphi_k} - \boldsymbol{\epsilon}^r \cdot \frac{\partial \mathbf{D}}{\partial \varphi_k} \cdot \boldsymbol{\epsilon}\right], \quad \delta\lambda H_1 = 0 \tag{62}$$

and similar relations for the stationarity of G_1'. The second equality (62) requires either $H_1 = 0$ or $\delta\lambda = 0$.

An alternative formulation of the optimal design problem would require the minimization of G_1 or G_1' with the upper bound set on the structure cost, thus

$$\min G \text{ subject to } \quad J \leq J_0. \tag{63}$$

Introducing the functional $\bar{G} = G_1 + \lambda(J - J_0)$, the conditions of stationarity of $\bar{G}$ are expressed as follows

$$\delta G_1 = -\lambda\delta J, \quad \delta\lambda(J - J_0) = 0 \tag{64}$$

and we obtain the relations equivalent to (62). Consider, for instance, the global compliance design when the complementary energy Π_σ of the structure is to be minimized. Combining (47) with (64), the optimality conditions take the form

$$\frac{\partial W}{\partial \varphi_k} = -\lambda\frac{\partial F}{\partial \varphi_k} \tag{65}$$

which was already discussed in [5]. The identification problems differ from the optimal design problems only in the absence of cost function which is usually not associated with the identification procedure.

Assume, for instance, that the displacement field $\mathbf{u}_m$ was measured over some control surface S_m/which may be a portion of S_T/. For specified shape of a structure, the problem is reduced to determining a set of functions or parameters which occur in the stiffness matrix, $\mathbf{D}=\mathbf{D}(\varphi_k)$, $k=1, 2,..K$, so that the distance between the measured and calculated displacements $\mathbf{u}_m$ and $\mathbf{u}$ over S_m is minimized. Let the measure of this distance be

$$G_1=\frac{1}{2}\int \alpha(\mathbf{u}-\mathbf{u}_m)^2\, \mathrm{d}S_m \rightarrow \min_{\varphi_k}. \tag{66}$$

Introducing the adjoint structure satisfying the boundary conditions

$$\mathbf{T}^a=\alpha(\mathbf{u}-\mathbf{u}_m) \text{ on } S_m, \quad \mathbf{T}^{a_0}=0 \text{ on } S_T, \quad \mathbf{u}^{a_0}=0 \text{ on } S_u, \tag{67}$$

the variation of G_1 is expressed as follows

$$\delta G_1=-\int \boldsymbol{\epsilon}^a\cdot\frac{\partial \mathbf{D}}{\partial\varphi_k}\cdot\boldsymbol{\epsilon}\delta\varphi_k\, \mathrm{d}V=\int \boldsymbol{\sigma}^a\cdot\frac{\partial \mathbf{E}}{\partial\varphi_k}\cdot\boldsymbol{\sigma}\delta\varphi_k\, \mathrm{d}V \tag{68}$$

and the condition $\delta G_1=0$ provides the necessary optimality conditions

$$\boldsymbol{\epsilon}^a\cdot\frac{\partial \mathbf{D}}{\partial\varphi_k}\cdot\boldsymbol{\epsilon}=0. \tag{69}$$

In particular, when the stiffness matrix is uniform and depends on a set of parameters φ_k to be identified, instead of (68), we obtain

$$\delta G_1=-\sum_{k=1}^{K}\delta\varphi_k\int \boldsymbol{\epsilon}^a\cdot\frac{\partial \mathbf{D}}{\partial\varphi_k}\cdot\boldsymbol{\epsilon}\, \mathrm{d}V \tag{70}$$

and the optimality conditions are

$$\int \boldsymbol{\epsilon}^a\cdot\frac{\partial \mathbf{D}}{\partial\varphi_k}\cdot\boldsymbol{\epsilon}\, \mathrm{d}V=0, \quad (k=1, 2,..K). \tag{71}$$

4. EXAMPLES

In this Section, three simple examples will be presented, where the optimality conditions discussed in the previous section are used and the variation of a stress functional is expressed in a particular case. However, the applicability of the obtained results is much broader. Moreover, they can be extended for the sensitivity analysis in the case of shape variation.

4.1 *Example* 1. *Optimal design of a beam in order to minimize its maximal deflection*

Consider a sandwich beam shown in Fig. 4 composed of I segments of sheet thickness t_1, $t_2, \ldots t_I$ and constant thickness of the core equals $2h$. The beam is simply supported at one end and built-in at the other end. Under specified lateral loading $T(x)$, and specified lengths of segments l_i, the design is aimed at determining the values of t_1, $t_2 \ldots t_I$ corresponding to minimal cost of the material and satisfying the constraint $u(x)\leqslant u_0$, for $x\epsilon\langle 0,L\rangle$, where $u(x)$ denotes the lateral displacement. Instead of local constraint, let us introduce the global displacement

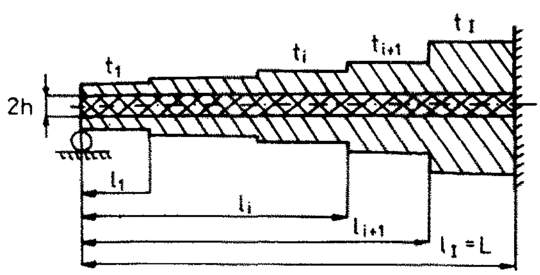

Fig. 4. Sandwich beam of I segments.

condition in the form

$$G = \int g(u)\,\mathrm{d}x = \int \left(\frac{u}{u_0}\right)^n \mathrm{d}x. \tag{72}$$

Note that for $n \to \infty$, $g(u) \to 0$ for $u < u_0$ and $g(u) \to \infty$ for $u \geqslant u_0$. Thus, for large values of n, the displacement exceeding the value u_0 will induce a large contribution to the global measure (72). The cost of the beam is expressed as follows

$$C = 2bc_0 \sum_{i=1}^{I} t_i(l_i - l_{i-1}), \tag{73}$$

where b denotes the beam width and c_0 is the specific material cost. Instead of setting the constraint $G \leqslant G_0$ and minimizing C, let us study the evolution of design with respect to uniform thickness design of the same material volume. We therefore formulate the problem as follows

$$\min G \text{ subject to } C = C_0. \tag{74}$$

Introducing the adjoint beam, its loading is specified as follows

$$T_i^a(x) = \frac{\partial}{\partial u_i}\left(\frac{u_i}{u_0}\right)^n = \frac{n}{u_0}\left(\frac{u_i}{u_0}\right)^{n-1}, \tag{75}$$

$$l_{i-1} \leqslant x \leqslant l_i, \quad i = 1,2,\ldots I,$$

where T_i^a denotes the loading over ith segment. Denoting by $M_i(x, t_k)$ the bending moment field in the ith segment and by $M_i^a(x, t_k)$ the bending moment of the adjoint beam, and noting that the compliance of each segment equals

$$E_i = \frac{1}{2Eh^2bt_i}, \quad i = 1,2\ldots I, \tag{76}$$

where E denotes the Young modulus, the necessary optimality conditions (62) are expressed in the form

$$\frac{1}{t_i^2}\int_{l_{i-1}}^{l_i} M_i^a(x, t_k) M_i(x, t_k)\,\mathrm{d}x = 4\lambda E h^2 b^2 (l_i - l_{i-1}) \tag{77}$$

for $i = 1,2,\ldots I$, and

$$\sum_{k=1}^{I} t_k(l_k - l_{k-1}) = \frac{C_0}{2bc_0}. \tag{78}$$

These conditions combined with the equilibrium equations, continuity conditions between particular segments and boundary conditions at $x = 0$ and $x = L$, provide a set of non-linear algebraic equations which were solved with respect to t_i and λ by using the Newton–Raphson procedure. The solution is illustrated for the case of a beam loaded by two concentrated forces and divided into four and eight equal segments, Fig. 5(a, b). The following Table 1 illustrates the evolution of design with n and the dependence of maximal deflection on n. Figure 5(c) shows the deflection field for the uniform design and optimal designs with four and eight segments and the same material volume. It is seen that for increasing n, the maximal deflection is gradually reduced.

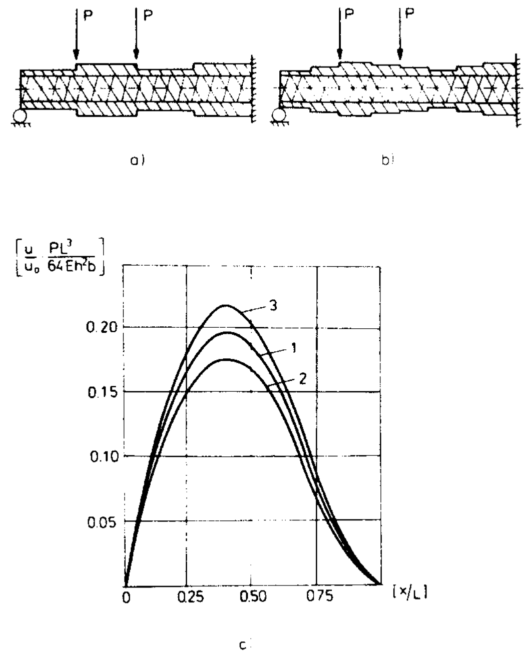

Fig. 5. Sandwich beam loaded by two forces and divided into four (a) and eight (b) equal segments; (c) Deflection field for optimal designs with four (1) and eight (2) segments and the uniform design (3) and the same material volume ($n = 4$).

Table 1. Optimal thickness and maximal deflections of a sandwich beam of constant volume

Exponent [n]	Optimal values of segment thickness t_i [cm]								Max. defl. $\left[\frac{uPL^3}{64u_0Eh^2b}\right]$
Uniform design									
-	2.5								0.2183
Four-segment design									
2	1.822		3.027		1.422		3.729		0.1985
3	1.816		3.100		1.457		3.627		0.1981
4	1.809		3.145		1.476		3.571		0.1980
Eight-segment design									
2	1.041	2.613	3.416	3.301	2.261	0.648	2.197	4.523	0.1786
3	1.018	2.611	3.479	3.390	2.311	0.657	2.153	4.381	0.1782
4	1.003	2.604	3.517	3.448	2.339	0.658	2.124	4.306	0.1780

4.2 *Example 2. Identification of beam compliances within specified segments*

Consider a cantilever beam of two equal segments of different compliances E_1, E_2 where

$$E_i = \frac{12}{Ebh^3}, \qquad i = 1, 2. \tag{79}$$

The measured deflection field due to concentrated load P at the beam end can be approximated by a parabolic function

$$u_m = \beta(x - L)^2, \tag{80}$$

where β is constant. It is our purpose to identify the values of compliances E_1, E_2 in order to minimize the squared 'distance' of predicted and measured deflections, thus

$$G(h_1,h_2)=\frac{1}{2}\int_0^L [u(x)-u_m(x)]^2\,dx \to \min_{h_1,h_2}. \tag{81}$$

Introducing the adjoint beam of the same support conditions and loaded by, see Fig. 6(a,b)

$$T^a(x)=u(x)-u_m(x), \tag{82}$$

the necessary optimality condition is expressed in the form

$$\delta G=\int_0^L M^a(x)\frac{\partial E_i}{\partial h_i}M(x)\delta h_i\,dx=0, \tag{83}$$

which leads to two equations

$$\int_0^{1/2L} M_1{}^a(x)M(x)\,dx=0,\quad \int_{1/2L}^{L} M_2{}^a(x)M(x)\,dx=0, \tag{84}$$

whose solution can be obtained in a closed form, namely

$$h_1=1.1925\sqrt[3]{\left(\frac{PL}{Eb\beta}\right)},\quad h_2=1.7141\sqrt[3]{\left(\frac{PL}{Eb\beta}\right)}. \tag{85}$$

Figure 6(c) shows the deflection field of the uniform beam having the deflection $u=\beta L^2$ at the tip $x=0$ and the optimally identified beam.

4.3 *Example* 3. *Variation of stress or strain functionals*

Consider a statically indeterminate beam, Fig. 7(a), loaded at its end by the bending moment M_b. Let the beam curvature and bending moment be denoted by x and M, whereas the beam stiffness be $\mathbf{D}=EJ$ and the compliance $\mathbf{E}=1/EJ$.

Consider the functional G in the form

$$G=\int \phi(x,J)\,dx=\int_0^L \frac{\kappa}{J}\,dx=\int_0^L \frac{M^3}{E^3J^4}\,dx=\int_0^L \psi(M,J)\,dx. \tag{86}$$

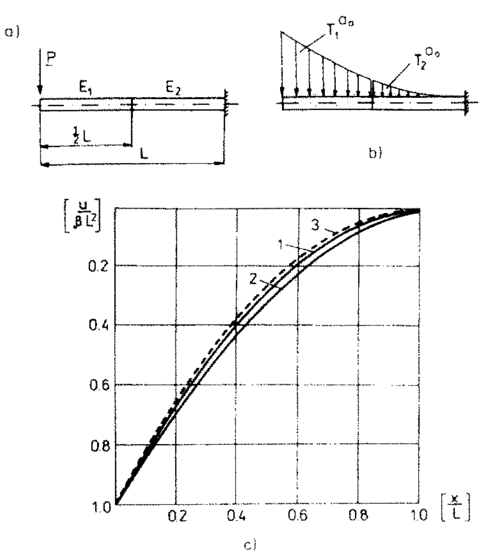

Fig. 6. Cantilever beam of two segments of different compliances; (a) Primary beam, (b) Adjoint beam, (c) Deflection field of the optimal beam (1) and uniform beam (2) compared with measured deflection (3).

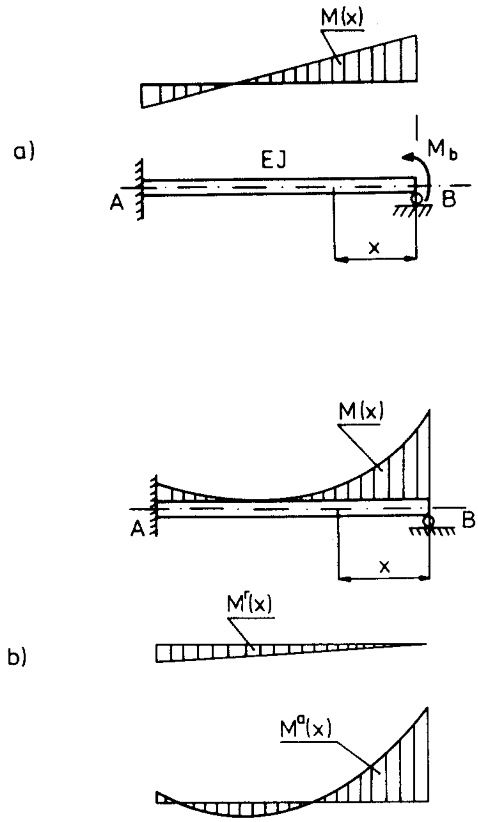

Fig. 7. (a) Beam loaded by the bending moment M_b, (b) Adjoint beam subjected to the initial moment field.

Let us derive the first and the second variations of G corresponding to the variation of beam stiffness by using the general relations derived in Section 2. The bending moment within the beam is expressed as follows

$$EJ\kappa = M = M_b\left(1-\frac{3}{2}\frac{x}{L}\right). \tag{87}$$

The adjoint beam is subjected to the initial moment field, see Fig. 7(b)

$$M^i = \frac{\partial\phi}{\partial\kappa} = 3\frac{\kappa^2}{J} = 3\frac{M_b^2}{E^2J^3}\left(1-\frac{3}{2}\frac{x}{L}\right)^2 \tag{88}$$

and

$$EJ\kappa^z = M^i + M^r = M^a = 3\frac{M_b^2}{E^2J^3}\left(1-\frac{3}{2}\frac{x}{L}\right)^2 + R_Bx \tag{89}$$

where R_B denotes the support reaction at B. Satisfying the boundary conditions at $x = 0$ and $x = L$, we find

$$R_B = -\frac{9}{16}\frac{M_b^2}{E^2J^3L}, \quad M^a = 3\frac{M_b^2}{E^2J^3}\left[\left(1-\frac{3}{2}\frac{x}{L}\right)^2 - \frac{3}{16}\frac{x}{L}\right]. \tag{90}$$

The first variation of (86) can now easily be calculated. Since

$$\frac{\partial\phi}{\partial J} = -\frac{\kappa^3}{J^2}, \quad \frac{\partial \mathrm{E}}{\partial J} = -\frac{1}{EJ^2}, \tag{91}$$

in view of (29), we have

$$\delta G = \int_0^L -\frac{4M_b^3}{E^3J^5}\left(1-\frac{3}{2}\frac{x}{L}\right)^3 \delta J\,\mathrm{d}x. \tag{92}$$

The second variation is now expressed as follows

$$\delta^2 G = \int_0^L \frac{20M_b^3}{E^3J^6}\left(1-\frac{3}{2}\frac{x}{L}\right)^3 \delta J^2\,\mathrm{d}x. \tag{93}$$

5. CONCLUDING REMARKS

The present paper generalizes the results of previous works[1–8] and provides a systematic variational approach to sensitivity analysis and optimal design for structures of fixed shape with varying material parameters. The analysis is limited to linearly elastic structures for which the concept of adjoint structure provides an effective tool in generating first and second variations of arbitrary volume or surface integrals. Only static problems are discussed. However, the generalization to dynamic problems, when the functionals are defined over space and time domains, can be obtained by following the present analysis.

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